# Exact expectation values of local fields in quantum sine-Gordon model

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### **Abstract**

We propose an explicit expression for vacuum expectation values  $\langle e^{ia\varphi} \rangle$  of the exponential fields in the sine-Gordon model. Our expression agrees both with semi-classical results in the sine-Gordon theory and with perturbative calculations in the Massive Thirring model. We use this expression to make new predictions about the large-distance asymptotic form of the two-point correlation function in the XXZ spin chain.

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Many 2D Quantum Field Theories (QFT) can be realized as Conformal Field Theories (CFT) perturbed by some relevant operators [1]. In these cases the correlation function of the local fields  $\mathcal{O}_a(x)$  of the perturbed theory

$$\langle \mathcal{O}_{a_1}(x_1)...\mathcal{O}_{a_N}(x_N) \rangle$$
 (1)

are defined in a formal way through the Conformal Perturbation Theory (CPT) as

$$Z(\mu)^{-1} \langle \mathcal{O}_{a_1}(x_1)...\mathcal{O}_{a_N}(x_N) e^{-\mu \int d^2x \,\Phi(x)} \rangle_{CFT},$$
 (2)

where

$$Z(\mu) = \langle e^{-\mu \int d^2 x \, \Phi(x)} \rangle_{CFT},$$

 $\Phi$  is the local field taken as the perturbation,  $\mu$  is the CPT expansion parameter and  $\langle \dots \rangle_{CFT}$  denote the expectation values in the original CFT. Of course, the expression (2) for the correlation functions (1) can not be taken literally. As is well known, the power series expansion in  $\mu$  generated by (2) is always plagued by infrared divergences, reflecting the fact that the correlation functions (1) contain non-analytic in  $\mu$  terms. However these non-analytic terms are rather the property of the vacuum  $|0\rangle$  in  $\langle \dots \rangle = \langle 0 | \dots | 0 \rangle / \langle 0 | 0 \rangle$  then of the local operators  $\mathcal{O}_a$ . The local fields are believed to satisfy the OPE algebra

$$\mathcal{O}_a(x) \, \mathcal{O}_b(y) \simeq \sum_c \, C_{ab}^{\,c}(x-y) \, \mathcal{O}_c(y) \,.$$
 (3)

By the very virtue of these relations the c-number coefficient functions C are not very sensitive to the long-range properties of the theory (in particular, the CPT for the coefficients C do not contain any infrared divergences) and therefore one expects that they can be expanded into a power series in  $\mu$  with finite radius of convergence. On the other hand, by successive application of (3) any correlation function (1) can be reduced down to the one-point functions  $\langle \mathcal{O}_a(x) \rangle$ . Simple dimensional analysis shows that these quantities are non-analytic in  $\mu$ 

$$\langle \mathcal{O}_a(x) \rangle = G_a \ \mu^{\frac{\Delta_a}{1-\Delta}} \ , \tag{4}$$

where  $\Delta_a$  are conformal dimensions of the fields  $\mathcal{O}_a$  and  $G_a$  are constants. In writing (4) we have assumed that the perturbing operator  $\Phi$  consists of a single scaling field of the conformal dimension  $\Delta$ . In this sense it is the one-point functions (4) that contain all the non-perturbative information about the QFT which can not be extracted directly from

CPT [2]. Therefore the problem of calculation of the one-point functions has fundamental significance <sup>1</sup>.

During the last two decades much progress has been made in understanding 2D Integrable QFT. However the closed expressions for the one-point functions are obtained in very few cases. Most significant result obtained in this direction so far was exact computation in many cases the expectation value of the perturbing operator  $\Phi$  in (2) with the help of Thermodynamic Bethe Ansatz and similar techniques [3]. In this paper we propose an explicit expression for the one-point functions of all exponential fields  $\mathcal{O}_a(x) = e^{ia\varphi}(x)$  in the sine-Gordon QFT.

The sine-Gordon model is defined by the Euclidean action

$$\mathcal{A}_{SG} = \int d^2x \left\{ \frac{1}{16\pi} (\partial_{\nu}\varphi)^2 - 2\mu \cos(\beta\varphi) \right\}, \tag{5}$$

where  $\beta$  and  $\mu$  are parameters <sup>2</sup>. The renormalization required with this definition is relatively simple. The constant  $\beta$  does not renormalize while  $\mu$  is renormalized multiplicatively. In order to give the constant  $\mu$  a precise meaning one has to fix the normalization of the field  $\cos(\beta\varphi)$ . A convenient normalization which we will accept here corresponds to the short distance limit of the two-point function

$$\langle \cos(\beta\varphi)(x) \cos(\beta\varphi)(y) \rangle \to \frac{1}{2} |x-y|^{-4\beta^2} \text{ as } |x-y| \to 0.$$
 (6)

This normalization is the most natural one if one understands (5) as a perturbed CFT, with the field  $\cos(\beta\varphi)$  taken as the perturbing operator. Under this normalization the field  $\cos(\beta\varphi)$  has the dimension  $\left[length\right]^{-2\beta^2}$  and correspondingly the dimension of the constant  $\mu$  is  $\left[length\right]^{2\beta^2-2}$ .

The theory (5) has a discrete symmetry  $\varphi \to \varphi + 2\pi n/\beta$  with any integer n. In this paper we study (5) in the domain  $0 < \beta^2 < 1$ , where the above symmetry is spontaneously broken (and the QFT (5) is massive), so that the theory has infinitely many ground states  $|0_n\rangle$  characterized by the associated expectation values of the field  $\varphi$ ,  $\langle \varphi \rangle_n = 2\pi n/\beta$ , where  $\langle \dots \rangle_n = \langle 0_n | \dots |0_n \rangle / \langle 0_n |0_n \rangle$ . In what follows we concentrate our attention on one of these ground states, say  $|0_0\rangle$ , and use the notation  $\langle \dots \rangle \equiv \langle \dots \rangle_0$ .

 $<sup>^{1}</sup>$  Although our discussion explicitly concerns 2D QFT, the general statements above are expected to hold in multidimensional QFT as well.

<sup>&</sup>lt;sup>2</sup> Note that our notation  $\beta$  for the sine-Gordon coupling constant differs by a factor  $\sqrt{8\pi}$  from conventional one [4].

The sine-Gordon model admits an equivalent description as the Massive Thirring model [4]

$$\mathcal{A}_{MT} = \int d^2x \left\{ i \bar{\psi} \gamma^{\nu} \partial_{\nu} \psi - \mathcal{M} \bar{\psi} \psi - \frac{g}{2} (\bar{\psi} \gamma^{\nu} \psi)^2 \right\}, \tag{7}$$

where  $\psi, \bar{\psi}$  is the Dirac field, the four-fermion coupling constant g relates to  $\beta$  in (5) as

$$\frac{g}{\pi} = \frac{1}{2\beta^2} - 1 \tag{8}$$

and  $\mathcal{M}$  is the mass parameter. In particular, the fermion current of (7) is related to the field  $\varphi$  in (5) as

$$j^{\nu} \equiv \bar{\psi}\gamma^{\nu}\psi = -\frac{\beta}{2\pi} \; \epsilon^{\nu\lambda} \, \partial_{\lambda}\varphi \; . \tag{9}$$

The QFT (5) is integrable and its on-shell solution, i.e. spectrum of particles and S-matrix, is well known [5]. The theory contains soliton S, antisoliton  $\bar{S}$  (these particles coincide with the "fundamental fermions" of the Lagrangian (7)) and a number of soliton-antisoliton bound states ("breathers")  $B_n$ ,  $n = 1, 2, ..., < 1/\xi$ ; here and below the notation

$$\xi = \frac{\beta^2}{1 - \beta^2} \tag{10}$$

is used. The lightest of these bound states  $B_1$  coincides with the particle associated with the field  $\varphi$  in perturbative treatment of the QFT (5). Its mass m is given by

$$m = 2M \sin\left(\pi\xi/2\right) \,, \tag{11}$$

where M is the soliton mass. Exact relation between the particle masses and the parameter  $\mu$  in (5) defined with respect to the normalization (6) was found in [3]

$$\mu = \frac{\Gamma(\beta^2)}{\pi \Gamma(1 - \beta^2)} \left[ M \frac{\sqrt{\pi} \Gamma\left(\frac{1}{2} + \frac{\xi}{2}\right)}{2\Gamma\left(\frac{\xi}{2}\right)} \right]^{2 - 2\beta^2}.$$
 (12)

This allows one to derived the vacuum expectation value of the field  $\exp(i\beta\varphi)$  using an obvious relation

$$\langle e^{i\beta\varphi} \rangle = -\frac{1}{2} \partial_{\mu} f(\mu) , \qquad (13)$$

where  $f(\mu)$  is the specific free energy  $f(\mu) = -\frac{1}{V} \log Z$ , which is also known exactly [6], [7]

$$f(\mu) = -\frac{M^2}{4} \tan\left(\pi\xi/2\right) \,. \tag{14}$$

Combining (11)-(14) one finds

$$\langle e^{i\beta\varphi} \rangle = \frac{(1+\xi)\pi\Gamma(1-\beta^2)}{16\sin(\pi\xi)\Gamma(\beta^2)} \left( \frac{\Gamma(\frac{1}{2} + \frac{\xi}{2})\Gamma(1-\frac{\xi}{2})}{4\sqrt{\pi}} \right)^{2\beta^2 - 2} m^{2\beta^2} , \qquad (15)$$

where we have assumed that the field  $e^{i\beta\varphi}$  is normalized according to (6).

In this paper we study the expectation values (4) of the exponential fields  $\mathcal{O}_a(x) = e^{ia\varphi}(x)$  in the sine-Gordon model. Like in (6), we assume that these fields are normalized in accordance with the following short distance limiting form of their two-point functions

$$\langle e^{ia\varphi}(x) e^{-ia\varphi}(y) \rangle \to |x-y|^{-4a^2}$$
 as  $|x-y| \to 0$ . (16)

Under this normalization the field  $e^{ia\varphi}$  has the dimension  $\left[length\right]^{-2a^2}$ .

For two special values of  $\beta^2$  the one-point function

$$\mathcal{G}_a = \langle e^{ia\varphi} \rangle \tag{17}$$

admits direct calculation.

a) The semi-classical limit  $\beta^2 \to 0$ . In Appendix A we study the expectation values (17) with  $a = \alpha/\beta$  for  $\beta^2 \to 0$  and fixed  $\alpha$ , and obtain the result

$$\langle e^{i\frac{\alpha}{\beta}\varphi}\rangle \to D(\alpha) \left(\frac{m}{4}\right)^{2\frac{\alpha^2}{\beta^2}} \exp\left\{\frac{1}{\beta^2} \int_0^\infty \frac{dt}{t} \left[\frac{\sinh^2(2\alpha t)}{t \sinh(2t)} - 2\alpha^2 e^{-2t}\right]\right\},$$
 (18)

where  $D(\alpha)$  can be calculated as the functional determinant (A.11).

b)  $\beta^2=1/2$ . In this case the Thirring coupling constant (8) vanishes and the sine-Gordon model is equivalent to the free-fermion theory. In Appendix B we use this simplification to obtain for  $\beta^2=1/2$  and  $|\Re e\,a|<1/\sqrt{2}$ 

$$\langle e^{ia\varphi} \rangle = \left(\frac{M}{2}\right)^{2a^2} \exp\left\{ \int_0^\infty \frac{dt}{t} \left[ \frac{\sinh^2(\sqrt{2at})}{\sinh^2(t)} - 2a^2 e^{-2t} \right] \right\}, \tag{19}$$

where M is the mass of the free fermion field.

Evident similarity between (18) and (19) suggests the following expression for the expectation value (17) for generic  $\beta^2 < 1$  and  $\left| \Re e \, a \, \right| < 1/(2\beta)$ 

$$\mathcal{G}_{a} = \left[ \frac{m \Gamma\left(\frac{1}{2} + \frac{\xi}{2}\right) \Gamma\left(1 - \frac{\xi}{2}\right)}{4\sqrt{\pi}} \right]^{2a^{2}} \times \exp\left\{ \int_{0}^{\infty} \frac{dt}{t} \left[ \frac{\sinh^{2}(2a\beta t)}{2\sinh(\beta^{2}t)\sinh(t)\cosh\left((1 - \beta^{2})t\right)} - 2a^{2}e^{-2t} \right] \right\}.$$
(20)

The pre-exponential factor in (20) is chosen to agree with (15). The formula (20) is our conjecture for (17).

The expression (20) can be expanded in power series in  $\beta^2$  or in g (8) and the coefficients of these expansions can be compared with the perturbative calculations for the actions (5) and (7), respectively. To perform this check it is more convenient to consider the expectation values  $\langle \varphi^{2n} \rangle$  which can be readily obtained by expanding (17) in power series in  $a^2$ . Let us define "fully connected" one-point functions  $\sigma_{2n}$  as

$$\langle e^{ia\varphi} \rangle \equiv 1 + \sum_{n=1}^{\infty} \frac{(-a^2)^n}{(2n)!} \langle \varphi^{2n} \rangle = \exp\left(\sum_{n=1}^{\infty} \frac{(-a^2)^n}{(2n)!} \sigma_{2n}\right),$$
 (21)

so that

$$\sigma_2 = \langle \varphi^2 \rangle; \qquad \sigma_4 = \langle \varphi^4 \rangle - 3 \langle \varphi^2 \rangle^2; \qquad \dots$$
 (22)

Expansion of (20) gives

$$\sigma_{2} = -4 \log \left\{ \frac{m \Gamma\left(\frac{1}{2} + \frac{\xi}{2}\right) \Gamma\left(1 - \frac{\xi}{2}\right)}{4\sqrt{\pi}} \right\} - 4 \int_{0}^{\infty} \frac{dt}{t} \left\{ \frac{\beta^{2} t^{2}}{\sinh(\beta^{2}t) \sinh(t) \cosh\left((1 - \beta^{2})t\right)} - e^{-2t} \right\},$$

$$\sigma_{2n} = (-1)^{n} 4^{2n-1} \beta^{2n} \int_{0}^{\infty} \frac{t^{2n-1} dt}{\sinh(\beta^{2}t) \sinh(t) \cosh\left((1 - \beta^{2})t\right)}, \quad n > 1.$$
(23)

We show in Appendix C that  $\sigma_2$  and  $\sigma_4$  in (23) agree with the perturbation theory for (5) up to  $\beta^4$ , and that  $\sigma_2$  agrees with the perturbation theory for (7) up to g.

Clearly, more checks of (20) are desirable. We note in this connection that the expectation value (17) controls both short and long distance asymptotics of the two-point correlation function

$$\mathcal{G}_{aa'}(|x-y|) = \langle e^{ia\varphi}(x) e^{ia'\varphi}(y) \rangle$$
 (24)

with  $|a + a'| < \beta/2$ . Indeed, if this inequality is satisfied the short distance limit of (24) is dominated by OPE

$$e^{ia\varphi}(x) e^{ia'\varphi}(y) \to |x-y|^{4aa'} e^{i(a+a')\varphi}(y)$$
 as  $|x-y| \to 0$ . (25)

Therefore

$$\mathcal{G}_{aa'}(r) \rightarrow \begin{cases}
\mathcal{G}_{a+a'} r^{4aa'} & \text{as} & r \to 0 \\
\mathcal{G}_a \mathcal{G}_{a'} & \text{as} & r \to \infty
\end{cases}$$
(26)

It is probably possible to check this relation by calculating numerically the correlation function (24) with the use of exact form-factors [8]. It is also worth noting that the expression (20) is expected to hold for the one-point functions of the sinh-Gordon model as well, if one makes the substitution  $\beta^2 \to -\beta^2$  in (20). The form-factors for the sinh-Gordon model can be found in [8], [9].

The one-point functions (17) of the sine-Gordon model can be used to derive the one-point functions of primary fields in c < 1 "minimal" CFT perturbed with the operator  $\Phi_{1,3}$ . As is known, the perturbed "minimal model"  $\mathcal{M}_{p/p'}$  (with  $c = 1 - \frac{6(p'-p)^2}{p \, p'}$ ) can be obtained by "quantum group restrictions" from the sine-Gordon QFT (5) with  $\xi = \frac{p}{p'-p}$  [10], [11]. Using this relation one obtains from (20)

$$\langle \Phi_{l,k} \rangle = \left[ M \frac{\sqrt{\pi} \Gamma\left(\frac{3}{2} + \frac{\xi}{2}\right)}{2 \Gamma\left(\frac{\xi}{2}\right)} \right]^{2\Delta_{l,k}} \mathcal{Q}\left((\xi + 1)l - \xi k\right) , \qquad (27)$$

where the function  $Q(\eta)$  for  $\big|\Re e\ \eta\,\big|<\xi$  is given by the integral

$$\mathcal{Q}(\eta) = \exp\left\{ \int_0^\infty \frac{dt}{t} \left( \frac{\cosh(2t) \sinh(t(\eta - 1)) \sinh(t(\eta + 1))}{2 \cosh(t) \sinh(t\xi) \sinh(t(1 + \xi))} - \frac{(\eta^2 - 1)}{2\xi(\xi + 1)} e^{-4t} \right) \right\}$$

and it is defined through analytic continuation outside this domain. In (27)  $\Phi_{l,k}$  stands for the primary field of the dimension

$$\Delta_{l,k} = \frac{\left((\xi+1)l - \xi k\right)^2 - 1}{4\xi(\xi+1)} \tag{28}$$

with canonical normalization

$$\langle \Phi_{l,k}(x) \Phi_{l,k}(y) \rangle \to |x-y|^{-4\Delta_{l,k}} \quad \text{as} \quad |x-y| \to 0 ,$$
 (29)

and M denotes physical mass in the perturbed theory.

Finally, we remark that accepting the conjecture (20) one can make an interesting prediction about long distance asymptotic of two-point correlation function in the XXZ spin chain

$$\frac{\langle vac \mid \sigma_s^x \sigma_{s+n}^x \mid vac \rangle}{\langle vac \mid vac \rangle} \to F(\beta^2) \ n^{-\beta^2} \quad \text{as} \quad n \to \infty , \tag{30}$$

where  $|vac\rangle$  is the ground state of the XXZ Hamiltonian <sup>3</sup>

$$\mathbf{H}_{XXZ} = -\frac{1 - \beta^2}{2\sin(\pi\beta^2)} \sum_{s=-\infty}^{\infty} \left( \sigma_s^x \sigma_{s+1}^x + \sigma_s^y \sigma_{s+1}^y + \cos(\pi\beta^2) (\sigma_s^z \sigma_{s+1}^z - 1) \right) , \qquad (31)$$

<sup>&</sup>lt;sup>3</sup> As usual, the infinite XXZ spin chain is defined as  $N \to \infty$  limit of a finite chain of size N with periodic boundary conditions.

 $\sigma_s^x, \sigma_s^y$  and  $\sigma_s^z$  are Pauli matrices associated with the site s of the chain. While the power-like behavior in (30) is well known, the factor

$$F(\beta^2) = \frac{(1+\xi)^2}{2} \left[ \frac{\Gamma(\frac{\xi}{2})}{2\sqrt{\pi} \Gamma(\frac{1}{2} + \frac{\xi}{2})} \right]^{\beta^2} \times \exp\left\{ -\int_0^\infty \frac{dt}{t} \left( \frac{\sinh(\beta^2 t)}{\sinh(t)\cosh((1-\beta^2)t)} - \beta^2 e^{-2t} \right) \right\}$$
(32)

in (30) is the consequence of (20) (see Appendix D).

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# Appendix A.

The expectation value (17) can be calculated as the Euclidean functional integral

$$\mathcal{G}_a = Z^{-1}(\mu) \int \left[ \mathcal{D}\varphi \right] e^{ia\varphi(0)} e^{-\mathcal{A}_{SG}} , \qquad (A.1)$$

where  $\mathcal{A}_{SG}$  is the action (5). For  $a = \alpha/\beta$ ,  $\beta \to 0$  this integral is dominated by a saddlepoint configuration  $\varphi(x) = \phi(x)/\beta$  which solves the classical equation

$$\partial_{\nu}^{2} \phi(x) = m^{2} \sin \phi(x) - 8\pi i \alpha \delta^{(2)}(x) ,$$
 (A.2)

and decays sufficiently fast at  $|x| \to \infty$ . Obviously, the solution depends only on the radial coordinate  $\phi = \phi(r)$ , r = |x|. Note that  $\phi$  is real only if  $\alpha$  is imaginary,  $\alpha = i\omega$ . The classical action calculated on this solution diverges. Proper treatment of this singularity requires introducing small cutoff distance  $\varepsilon$  and taking the limit

$$\log \langle e^{-\frac{\omega}{\beta}\varphi(0)} \rangle \sim \frac{1}{\beta^2} \lim_{\varepsilon \to 0} \left( \omega^2 \log(\varepsilon^2) - \omega \, \phi(\varepsilon) - 2 \, S(\omega, \varepsilon) \right) \,, \tag{A.3}$$

where

$$S(\omega, \varepsilon) = \int_{\varepsilon}^{\infty} \frac{r dr}{16} \left\{ (\partial_r \phi)^2 + 2m^2 \left( 1 - \cos \phi \right) \right\}, \tag{A.4}$$

and  $\phi(r)$  is the solution to the equation

$$\partial_r^2 \phi + r^{-1} \,\partial_r \phi - m^2 \sin \phi = 0 \,\,\,\,(A.5)$$

which satisfies the asymptotic conditions

$$\phi(r) \to 4\omega \left[ \log \left( \frac{mr}{2} \right) + C_{\omega} \right] \quad \text{as} \quad r \to 0 ,$$
 (A.6)

$$\phi(r) \to -\frac{4}{\pi} \sinh(\pi\omega) \ K_0(mr) \quad \text{as} \quad r \to \infty \ .$$
 (A.7)

Here  $C_{\omega}$  is certain constant and  $K_0(t)$  is the Macdonald function. Let us introduce the function

$$S(\omega) = \lim_{\varepsilon \to 0} \left( \omega^2 \log \left( \frac{m\varepsilon}{2} \right) + S(\omega, \varepsilon) \right), \tag{A.8}$$

Taking derivative of (A.3) with respect to  $\omega$  and using the equation (A.5) one can derive the relation

$$\log \langle e^{-\frac{\omega}{\beta}\varphi(0)} \rangle \sim \frac{2}{\beta^2} \left( -\omega^2 \log \left( \frac{m}{2} \right) + \omega \int_0^\omega \frac{d\tau}{\tau^2} S(\tau) \right). \tag{A.9}$$

As is shown in Appendix B the function (A.8) admits the following representation

$$S(\omega) = \omega^2 - \int_0^\infty \frac{d\nu}{2\pi} \log \left( \frac{\cosh^2 \pi (\nu - \omega) \cosh^2 \pi (\nu + \omega)}{\cosh^2 \pi \nu \cosh \pi (\nu - 2\omega) \cosh \pi (\nu + 2\omega)} \right) \Re e \, \Psi \left( \frac{1}{2} - i\nu \right) , \text{ (A.10)}$$

where  $\Psi(t) = \Gamma'(t)/\Gamma(t)$ . The exponential factor in (18) is derived from (A.9) and (A.10) with the use of the integral representation for  $\Psi(t)$ . The pre-exponential factor in (18) can be obtained by evaluating the functional integral (A.1) in the Gaussian approximation around the above classical solution  $\phi$ 

$$D(i\omega) = \left(\frac{m\varepsilon}{2}e^{\gamma}\right)^{2\omega^2} \left[ \text{Det}'\left(\frac{-\partial_{\mu}^2 + m^2\cos\phi}{-\partial_{\mu}^2 + m^2}\right) \right]^{-\frac{1}{2}}, \tag{A.11}$$

where  $\gamma = 0.577216...$  is the Euler constant. The first factor in (A.11) appeared because of the mass renormalization in (A.9); its main role is to cancel ultraviolet divergences in the determinant. Note that validity of (20) requires the determinant (A.11) to be

$$D(i\omega) = 2^{2\omega^2} \exp\left(-\frac{1}{2} \int_0^\infty \frac{dt}{t} \frac{\sin^2(2\omega t)}{\cosh^2 t}\right). \tag{A.12}$$

It would be very interesting to evaluate the functional determinant (A.11) directly and check (A.12).

# Appendix B.

The one-point function  $\langle e^{ia\varphi} \rangle$  can be expressed in terms of the appropriately regularized (see below) Euclidean functional integral

$$\mathcal{I}(a) = \int_{\mathcal{F}_a} \left[ \mathcal{D}\psi \mathcal{D}\bar{\psi} \right] e^{-\mathcal{A}_{MTM}} , \qquad (B.1)$$

where  $\mathcal{A}_{MTM}$  is the Euclidean version of the action (7) and the integration is taken over the space  $\mathcal{F}_a$  of "twisted" field configurations, such that  $\psi(x)$  and  $\bar{\psi}(x)$  acquire phases

$$\psi \to e^{i\frac{2\pi a}{\beta}}\psi, \qquad \bar{\psi} \to e^{-i\frac{2\pi a}{\beta}}\bar{\psi},$$
 (B.2)

when continued around the point x = 0<sup>4</sup>. In general case it is not known how one can evaluate the functional integral (B.1) directly. In the case  $\beta^2 = 1/2$  the action (7) becomes quadratic and the task simplifies drastically.

The problem is most conveniently treated in conformal polar coordinates  $(\eta, \theta)$ 

$$z = x + iy = e^{\eta + i\theta}$$
,  $\bar{z} = x - iy = e^{\eta - i\theta}$ . (B.3)

In this coordinates the action (7) with g = 0 takes the form

$$\mathcal{A}_{FF} = i \int_{0}^{2\pi} d\theta \int_{-\infty}^{\infty} d\eta \left\{ \Psi_{L}^{\dagger} (\partial_{\theta} - i\partial_{\eta}) \Psi_{L} + \Psi_{R}^{\dagger} (\partial_{\theta} + i\partial_{\eta}) \Psi_{R} - iM e^{\eta} \left( \Psi_{L}^{\dagger} \Psi_{R} + \Psi_{L} \Psi_{R}^{\dagger} \right) \right\}, \tag{B.4}$$

where  $\Psi_{L,R}(\eta,\theta)$  and  $\Psi_{L,R}^{\dagger}(\eta,\theta)$  are the components of the Dirac bi-spinors  $\psi$  and  $\bar{\psi}$  transformed to the coordinates (B.3)<sup>5</sup>. According to (B.2) they satisfy the quasi-periodicity conditions

$$\Psi_{L,R}(\eta, \theta + 2\pi) = -e^{2\sqrt{2}\pi i a} \Psi_{L,R}(\eta, \theta), \qquad \Psi_{L,R}^{\dagger}(\eta, \theta + 2\pi) = -e^{-2\sqrt{2}\pi i a} \Psi_{L,R}(\eta, \theta) .$$
(B.5)

In other words, the fields  $\psi, \bar{\psi}$  are defined on the universal cover of the punctured Euclidean plane  $\mathbb{R}^2/\{0\}$  and satisfy there the quasi-periodicity condition (B.2).

 $<sup>^{5}</sup>$   $\psi_{L}(x,y) = e^{-\frac{1}{2}\eta - \frac{i}{2}\theta}\Psi_{L}(\eta,\theta), \ \psi_{R}(x,y) = e^{-\frac{1}{2}\eta + \frac{i}{2}\theta}\Psi_{R}(\eta,\theta), \text{ where } \psi_{L,R} \text{ are the components}$  of the bi-spinor  $\psi$  in Cartesian coordinates x,y, and similarly for  $\Psi^{\dagger}$ .

As usual, the most efficient way to evaluate the functional integral (B.1) is to use the Hamiltonian formalism. The Hamiltonian picture which appears most natural in our case is the one where the polar angle  $\theta$  is treated as (Euclidean) time, and the Hilbert space  $\mathcal{H}$  is associated with the "equal time" slice  $\theta = const$ . We call this approach angular quantization<sup>6</sup>. In this Hamiltonian formalism the fields  $\Psi, \Psi^{\dagger}$  become operators acting in  $\mathcal{H}$ . They satisfy canonical equal-time anti-commutation relations

$$\{\Psi_L(\eta), \Psi_L^{\dagger}(\eta')\} = \delta(\eta - \eta') , \qquad \{\Psi_R(\eta), \Psi_R^{\dagger}(\eta')\} = \delta(\eta - \eta') , \qquad (B.6)$$

while the angular Hamiltonian derived from (B.4) has the form

$$\mathbf{K} = i \int_{-\infty}^{\infty} d\eta \left\{ -\Psi_L^{\dagger} \partial_{\eta} \Psi_L + \Psi_R^{\dagger} \partial_{\eta} \Psi_R - M e^{\eta} \left( \Psi_L^{\dagger} \Psi_R + \Psi_L \Psi_R^{\dagger} \right) \right\}.$$
 (B.7)

The  $\eta$ -dependence of the mass term here has the effect of preventing the fermions from penetrating too far in the positive  $\eta$  direction. We call this effect the "mass barrier". Correspondingly, the Hamiltonian (B.7) is diagonalized by the following decompositions

$$\Psi_{L}(\eta,\theta) = \int_{-\infty}^{\infty} \frac{d\nu}{\sqrt{2\pi}} c_{\nu} u_{\nu}(\eta) e^{-\nu\theta} , \qquad \Psi_{R}(\eta,\theta) = \int_{-\infty}^{\infty} \frac{d\nu}{\sqrt{2\pi}} c_{\nu} v_{\nu}(\eta) e^{-\nu\theta} , 
\Psi_{L}^{\dagger}(\eta,\theta) = \int_{-\infty}^{\infty} \frac{d\nu}{\sqrt{2\pi}} c_{\nu}^{\dagger} u_{\nu}^{*}(\eta) e^{\nu\theta} , \qquad \Psi_{R}(\eta,\theta) = \int_{-\infty}^{\infty} \frac{d\nu}{\sqrt{2\pi}} c_{\nu}^{\dagger} v_{\nu}^{*}(\eta) e^{\nu\theta}$$
(B.8)

in terms of the partial waves

$$\begin{pmatrix} u_{\nu}(\eta) \\ v_{\nu}(\eta) \end{pmatrix} = \frac{\sqrt{2M} e^{\frac{\eta}{2}}}{\Gamma(\frac{1}{2} - i\nu)} \left( \frac{M}{2} \right)^{-i\nu} \begin{pmatrix} K_{\frac{1}{2} - i\nu} (Me^{\eta}) \\ K_{\frac{1}{2} + i\nu} (Me^{\eta}) \end{pmatrix}, \tag{B.9}$$

which describe the fermion scattering off the "mass barrier"

$$\begin{pmatrix} u_{\nu}(\eta) \\ v_{\nu}(\eta) \end{pmatrix} \to \begin{pmatrix} 1 \\ 0 \end{pmatrix} e^{i\nu\eta} + S_F(\nu) \begin{pmatrix} 0 \\ 1 \end{pmatrix} e^{-i\nu\eta} \quad \text{as} \quad \eta \to -\infty . \tag{B.10}$$

Here

$$S_F(\nu) = \left(\frac{M}{2}\right)^{-2i\nu} \frac{\Gamma(\frac{1}{2} + i\nu)}{\Gamma(\frac{1}{2} - i\nu)}$$
(B.11)

<sup>&</sup>lt;sup>6</sup> Angular quantization can be regarded as QFT version of Baxter's Corner Transfer Matrix approach [6]. Angular quantization was previously used in [12] in calculation the sine-Gordon form-factors and correlation functions of Jost operators.

is the S-matrix associated with this process. In (B.8)  $c_{\nu}$  and  $c_{\nu}^{\dagger}$  are operators satisfying the anti-commutation relations

$$\{c_{\nu}, c_{\nu'}\} = \{c_{\nu}^{\dagger}, c_{\nu'}^{\dagger}\} = 0 ,$$
  
$$\{c_{\nu}, c_{\nu'}^{\dagger}\} = \delta(\nu - \nu') .$$
 (B.12)

The Hilbert space  $\mathcal{H}$  is the fermionic Fock space over the algebra (B.12) with the vacuum state  $|v\rangle$  which satisfies the equations

$$c_{\nu} | v \rangle = 0$$
 for  $\nu > 0$ ,  
 $c_{\nu}^{\dagger} | v \rangle = 0$  for  $\nu < 0$ . (B.13)

The angular Hamiltonian (B.7) can be written as

$$\mathbf{K} = \int_0^\infty d\nu \, \nu \left( c_{\nu}^{\dagger} c_{\nu} + c_{-\nu} c_{-\nu}^{\dagger} \right) \,. \tag{B.14}$$

As the "Euclidean time"  $\theta$  in (B.4) is compactified the functional integral (B.1) (with g=0) is given by the trace

$$\mathcal{I}(a) = \text{Tr}_{\mathcal{H}} \left[ e^{-2\pi \mathbf{K} + i\frac{2\pi a}{\beta}} \mathbf{Q} \right], \qquad (B.15)$$

where the the fermion charge

$$\mathbf{Q} = \int_{-\infty}^{\infty} d\eta \, \left( \Psi_L^{\dagger} \Psi_L + \Psi_R^{\dagger} \Psi_R \right) = \int_{0}^{\infty} d\nu \, \left( c_{\nu}^{\dagger} c_{\nu} + c_{-\nu} c_{-\nu}^{\dagger} \right)$$
 (B.16)

is introduced in order to impose the twisted boundary conditions (B.2).

The trace (B.15) requires regularization. One can define the system (B.4) as the limiting case  $\varepsilon \to 0$  of similar system in a semi-infinite box  $\eta \in [\log \varepsilon, \infty)^7$  with the boundary conditions

$$\left[\Psi_L(\eta) - \Psi_R(\eta)\right]_{\eta = \log \varepsilon} = \left[\Psi_L^{\dagger}(\eta) - \Psi_R^{\dagger}(\eta)\right]_{\eta = \log \varepsilon} = 0.$$
 (B.17)

Let us denote  $\mathcal{I}_{\varepsilon}(a)$  the trace (B.15) regularized this way. Simple analysis (which takes into account the form of the boundary state associated with the boundary conditions (B.17) in conformal case M=0) shows that

$$\langle e^{ia\varphi} \rangle = \lim_{\varepsilon \to 0} \varepsilon^{-2a^2} \mathcal{I}_{\varepsilon}(a) / \mathcal{I}_{\varepsilon}(0) .$$
 (B.18)

<sup>&</sup>lt;sup>7</sup> This corresponds to cutting out a small disc of the size  $\varepsilon$  around the puncture x=0 in the functional integral (B.1). The other infinity  $\eta \to +\infty$  does not cause any problem because of the "mass barrier".

Using the above regularization one can directly evaluate (B.15), (B.18), with the result<sup>8</sup>

$$\langle e^{ia\varphi} \rangle = \exp \left\{ \int_0^\infty \frac{d\nu}{2\pi i} \log \left( \frac{\left(1 + e^{-2\pi\nu + 2\sqrt{2}i\pi a}\right) \left(1 + e^{-2\pi\nu - 2\sqrt{2}i\pi a}\right)}{\left(1 + e^{-2\pi\nu}\right)^2} \right) \partial_\nu \log S_F(\nu) \right\},$$
(B.19)

where  $S_F(\nu)$  is the S-matrix (B.11). Using the integral representation

$$\frac{1}{i}\log S_F(\nu) = -2\nu\log\left(\frac{M}{2}\right) - \int_0^\infty \frac{dt}{t} \left[\frac{\sin(2\nu t)}{\sinh t} - 2\nu e^{-2t}\right]$$
(B.20)

one arrives at (19).

Eq. (19) can be checked against exact result of [14] for the spontaneous magnetization in the Ising model. Indeed, at the free-fermion point the operator  $\cos(\frac{\beta}{2}\varphi)$  can be expressed in term of two independent Ising spin fields  $\sigma^{(i)}(x)$  (i=1,2) as

$$\cos\left(\frac{\beta}{2}\varphi(x)\right)\Big|_{\beta^2 = \frac{1}{2}} = 2^{-\frac{1}{2}} \sigma^{(1)}(x) \sigma^{(2)}(x) , \qquad (B.21)$$

where  $\sigma^{(i)}(x)$  are normalized as follows

$$\sigma^{(i)}(x) \, \sigma^{(i)}(0) \to |x|^{-1/4} \,, \quad |x| \to 0 \,.$$
 (B.22)

According to [14]

$$\langle \sigma(0) \rangle^2 = M^{\frac{1}{4}} 2^{\frac{1}{6}} e^{-\frac{1}{4}} A^3 ,$$
 (B.23)

where A is the Glaisher constant. This constant can be expressed in term of the derivative of the Riemann zeta function,

$$A = \exp\left(-\zeta'(-1) + \frac{1}{12}\right) = 1.28242712910062...$$
 (B.24)

Evaluating the integral in (19) for  $a = 2^{-\frac{3}{2}}$  and using (B.24) one finds complete agreement with (B.23).

It is instructive to compare our one-point function (19) with known exact results for the two-point function (24) in the free-fermion theory [15], [16]. For  $\beta^2 = 1/2$  and aa' > 0(24) can be written as

$$\langle e^{ia\varphi}(x) e^{ia'\varphi}(0) \rangle = \langle e^{ia\varphi} \rangle \langle e^{ia'\varphi} \rangle \exp(\Sigma_{aa'}(M|x|/2)),$$
 (B.25)

 $<sup>^8\,</sup>$  This result was also obtained by Al. Zamolodchikov [13].

with  $\Sigma_{aa'}(\tau)$  denoting the integral

$$\Sigma_{aa'}(\tau) = -\frac{1}{2} \int_{\tau}^{\infty} \rho d\rho \left\{ (\partial_{\rho} \chi)^2 - 4 \sinh^2 \chi - \frac{2(a - a')^2}{\rho^2} \tanh \chi \right\}.$$
 (B.26)

Here  $\chi(\rho)$  is the solution of the differential equation

$$\partial_{\rho}^{2}\chi + \rho^{-1}\partial_{\rho}\chi = 2\sinh(2\chi) + \frac{2(a-a')^{2}}{\rho^{2}}\tanh\chi\left(1-\tanh^{2}\chi\right),\tag{B.27}$$

subject to the asymptotic conditions

$$\chi(\rho) \to \sqrt{2} (a + a') \left[ \log(\rho) + C_{aa'} \right], \quad \text{as} \quad \rho \to 0,$$

$$\chi(\rho) \to -\frac{2}{\pi} \left( \sin(\sqrt{2\pi} a) \sin(\sqrt{2\pi} a') \right)^{\frac{1}{2}} K_{\sqrt{2}(a-a')}(2\rho), \quad \text{as} \quad \rho \to \infty,$$
(B.28)

and  $C_{aa'}$  is a certain constant. Now, one can take the limit  $|x| \to 0$  in (B.25) and use (25) to obtain the relation

$$\Sigma_{aa'} \equiv \lim_{\tau \to 0} \left( -4aa' \log \tau + \Sigma_{aa'}(\tau) \right) = \log \left( \left( \frac{M}{2} \right)^{-4aa'} \frac{\mathcal{G}_{a+a'}}{\mathcal{G}_a \mathcal{G}_{a'}} \right), \tag{B.29}$$

where  $\mathcal{G}_a$  is the one-point function (B.19) and

$$aa' > 0, \quad |a + a'| < 2^{-\frac{3}{2}}.$$

We have checked the relation (B.29) numerically. Note that for pure imaginary  $a=i\,\omega/\sqrt{2}$ ,  $a'=i\,\omega'/\sqrt{2}$  the function  $\chi(\rho)$  also becomes pure imaginary. If in addition  $\omega'=\omega$ , then

$$\phi(r) = -2i\,\chi(mr/2)$$

satisfies (A.5), (A.7) in Appendix A, and hence

$$S(\omega) = \frac{1}{2} \left( \left. \Sigma_{aa'} - 4 \int_0^\infty d\rho \, \rho \, \sinh^2 \chi \, \right) \right|_{a=a'=i \, \omega/\sqrt{2}}, \tag{B.30}$$

where  $S(\omega)$  is defined by (A.8). The integral in (B.30) can be evaluated by observing that

$$4\rho \sinh^2 \chi = \frac{1}{2} \partial_{\rho} \left[ \rho^2 \left( 4 \sinh^2 \chi - (\partial_{\rho} \chi)^2 \right) \right]. \tag{B.31}$$

Using (B.29), (B.19) one arrives at (A.10).

# Appendix C.

Expanding (23) in power series in  $\beta^2$  one finds

$$\sigma_2 = -4 \left( \gamma + \log(m/2) \right) + \left( \zeta(3) - \pi^2 \right) \frac{\beta^4}{3} + O(\beta^6) ,$$
 (C.1)

$$\sigma_4 = 56 \zeta(3) \beta^2 + 72 \zeta(3) \beta^4 + O(\beta^6),$$
 (C.2)

where  $\zeta(s)$  is the Riemann zeta function and  $\gamma$  is the Euler constant. These expansions are to be compared with the results of perturbation theory for the action (5). In the perturbative calculations it is more convenient to use another (equivalent to (21)) definition of the "fully connected" one-point functions  $\sigma_{2n}$ 

$$\sigma_2 = \langle \varphi^2 \rangle = \lim_{\varepsilon \to 0} \left( \langle \varphi(x)\varphi(0) \rangle \Big|_{|x|=\varepsilon} + 4\log \varepsilon \right) , \qquad (C.3)$$

$$\sigma_{2n} = \langle \varphi(x_1)\varphi(x_2)\dots\varphi(x_{2n})\rangle_c\big|_{x_1 = x_2 = \dots = x_{2n}}, \qquad (C.4)$$

where  $\langle ... \rangle_c$  in (C.4) is connected 2n-point correlation function. The Feynman diagrams contributing to  $\sigma_2$  and  $\sigma_4$  up to the order  $\beta^4$  are shown in Fig.1. The calculations are straightforward<sup>9</sup> and the result coincides with (C.1), (C.2).

Expansion of  $\sigma_2$  (23) around  $\beta^2 = 1/2$  has the form

$$\sigma_2 = -4\left(1 + \gamma + \log(M/2)\right) + \frac{g}{\pi}\left(7\zeta(3) - 2\right) + O(g^2) , \qquad (C.5)$$

where  $g = \pi (1/2 - \beta^2)/\beta^2$ . The expectation value  $\langle \varphi^2 \rangle$  can be calculated in perturbation theory of the Massive Thirring model (7) since (9) allows one to relate the two-point function  $\langle \varphi(x)\varphi(y) \rangle$  in (C.3) to the current-current correlation function

$$\langle j^{\mu}(x) j^{\nu}(0) \rangle = \int \frac{d^2k}{(2\pi)^2} \left( \delta^{\mu\nu} - \frac{k^{\mu}k^{\nu}}{k^2} \right) \Pi(k^2) e^{ikx} ,$$
 (C.6)

$$\left(\beta/2\pi\right)^2 \left\langle \varphi(x)\varphi(0)\right\rangle = \int \frac{d^2k}{(2\pi)^2} \Pi(k^2) e^{ikx} . \tag{C.7}$$

The first two terms of the perturbative expansion in g for the function  $\Pi(k^2)$  can be obtained by calculating the contributions of the Feynman diagrams in Fig.2

$$M^{2}\Pi(k^{2}) = \frac{1}{4\pi} \frac{\sinh \theta - \theta}{\sinh \theta \sinh^{2}(\theta/2)} - \frac{g}{4\pi^{2}} \left[ \frac{(\sinh \theta - \theta)^{2}}{\sinh^{2} \theta \sinh^{2}(\theta/2)} - \frac{\theta^{2}}{\sinh^{2} \theta} \right] + O(g^{2}), \quad (C.8)$$

<sup>&</sup>lt;sup>9</sup> One only has to be sure to express the result in terms of the physical mass m.

where  $\theta$  is related to  $k^2$  as

$$k^2 = 4 M^2 \sinh^2(\theta/2) .$$

Evaluating the Fourier transform (C.7) and taking the limit (C.3) one obtains exactly (C.5).

### Appendix D.

Exact integral representation for the correlation function

$$\frac{\langle vac \mid \sigma_s^x \sigma_{s+n}^x \mid vac \rangle}{\langle vac \mid vac \rangle} \tag{D.1}$$

was found recently by Jimbo and Miwa [17]. However in this representation the correlation function (D.1) is expressed in terms of n-fold integral and therefore finding  $n \to \infty$  asymptotic form of (D.1) remains a challenging problem. Here we will explain how the asymptotic formula (30), (32) follows from our conjecture (20).

The XXZ model is a limiting case of XYZ spin chain

$$\mathbf{H}_{XYZ} = -\frac{1}{2\varepsilon} \sum_{s=-\infty}^{\infty} \left( J_x \, \sigma_s^x \, \sigma_{s+1}^x + J_y \, \sigma_s^y \, \sigma_{s+1}^y + J_z \, \sigma_s^z \, \sigma_{s+1}^z - J \right) , \tag{D.2}$$

with  $J_x \geq J_y \geq |J_z|$ ; we introduced here an auxiliary parameter  $\varepsilon$  which will be interpreted as a lattice spacing. Following Baxter [6], we use the parameterization

$$J_{x} = \frac{1 - \beta^{2}}{\pi} \left( \frac{\theta_{4}(\beta^{2})\theta'_{1}(0)}{\theta_{4}(\beta^{2})\theta_{1}(\beta^{2})} + \frac{\theta_{1}(\beta^{2})\theta'_{1}(0)}{\theta_{4}(\beta^{2})\theta_{4}(\beta^{2})} \right) ,$$

$$J_{y} = \frac{1 - \beta^{2}}{\pi} \left( \frac{\theta_{4}(\beta^{2})\theta'_{1}(0)}{\theta_{4}(\beta^{2})\theta_{1}(\beta^{2})} - \frac{\theta_{1}(\beta^{2})\theta'_{1}(0)}{\theta_{4}(\beta^{2})\theta_{4}(\beta^{2})} \right) ,$$

$$J_{z} = \frac{1 - \beta^{2}}{\pi} \left( \frac{\theta'_{1}(\beta^{2})}{\theta_{1}(\beta^{2})} - \frac{\theta'_{4}(\beta^{2})}{\theta_{4}(\beta^{2})} \right) ,$$

$$J = -\frac{1 - \beta^{2}}{\pi} \left( \frac{\theta'_{1}(\beta^{2})}{\theta_{1}(\beta^{2})} + \frac{\theta'_{4}(\beta^{2})}{\theta_{4}(\beta^{2})} \right) .$$
(D.3)

Here

$$\theta_1(v) = 2p^{\frac{1}{4}} \sin(\pi v) \prod_{n=1}^{\infty} (1 - p^{2n}) (1 - e^{2\pi i v} p^{2n}) (1 - e^{-2\pi i v} p^{2n}) ,$$

$$\theta_4(v) = \prod_{n=1}^{\infty} (1 - p^{2n}) (1 - e^{2\pi i v} p^{2n-1}) (1 - e^{-2\pi i v} p^{2n-1})$$

and the prime in (D.3) means a derivative. Note that for  $p \to 0$  the difference  $J_x - J_y$  vanishes and the XYZ model reduces to the XXZ chain (31). As is known [18], the XXZ model (31) describes the critical behavior of XYZ chain, i.e. the correlation length  $R_c = \varepsilon N_c \to \infty$  as  $p \to 0$ . The scaling limit  $p \to 0$ ,  $\varepsilon \sim p^{\frac{1}{4(1-\beta^2)}} \to 0$  of XYZ model is described by sine-Gordon QFT, and the following relation between the operators holds

$$\sigma_s^x \to N(\beta^2) \ \varepsilon^{\frac{\beta^2}{2}} \cos\left(\frac{\beta}{2}\varphi(r)\right), \qquad s\varepsilon \to r,$$
 (D.4)

where  $N(\beta^2)$  is normalization factor. The dependence of n in (30) is of course the simple consequence of (D.4) while the factor F in (30) is determined by the constant N,  $F = \frac{1}{2}N^2$ . It is exact value of this constant which can be deduced from (20) with the use of two exact results in XYZ model:

a) Exact formula for the energy gap due to Johnson, Krinski and McCoy [19], which gives in the limit  $p \to 0$  exact relation between p and the physical mass M in the sine-Gordon model [18]

$$M = \frac{4}{\varepsilon} p^{\frac{1+\xi}{4}} , \qquad (D.5)$$

where the notation (10) is used.

b) Exact result of Baxter and Kelland for the expectation value of the spin operator  $\sigma_s^x$  [20], which in the limit  $p \to 0$  reduces to

$$\frac{\langle vac \mid \sigma_s^x \mid vac \rangle}{\langle vac \mid vac \rangle} \rightarrow (1+\xi) \ p^{\frac{\xi}{8}} \ . \tag{D.6}$$

Comparing (D.6) with (20) and using (D.4), (D.5) one derives (32).

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# **FIGURES**

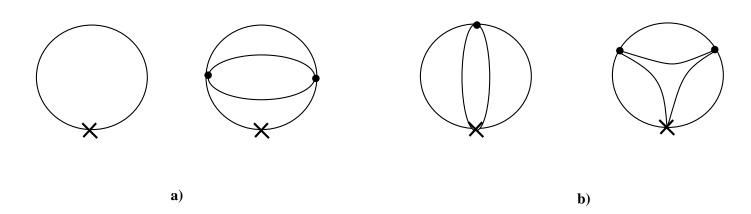


FIG.1

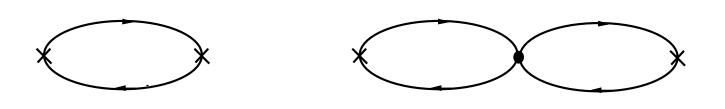


FIG.2